

LINEARNE DIFERENCNE JEDNAČINE

1. Ako je

- a) $a_1 = 5$, $a_2 = 7$ i $a_{n+2} = 2a_{n+1} - a_{n-1}$, $n \geq 0$;
 - b) $a_1 = 0$ i $a_n = 3 - 2a_{n-1}$, $n \geq 2$
 - c) $a_1 = 2$ i $a_{n+1} = 7 + 4a_n$, $n \geq 1$
- odrediti opšti član niza u funkciji od n .

► ARITMETIČKI NIZ

$$a_{n+1} = a_n + d \Rightarrow a_n = a_1 + (n-1)d = \underbrace{(a_1 - d)}_{\text{const}} + nd = c + d \cdot n$$

\Downarrow

$$a_n = c + n \cdot d \quad \text{je opšte rešenje}$$

► GEOMETRIJSKI NIZ

$$a_{n+1} = q \cdot a_n \Rightarrow a_n = a_1 \cdot q^{n-1} = \underbrace{\frac{a_1}{q}}_{\text{const}} q^n = k \cdot q^n$$

\Downarrow

$$a_n = k \cdot q^n \quad \text{je opšte rešenje}$$

► OPŠTA LINEARNA DIFERENCNA JEDNAČINA PRVOG REDA SA KONSTANTNIM KOEFICIJENTIMA

$$\begin{aligned} \blacklozenge: \quad & a_{n+1} = q \cdot a_n + d \\ & \left(\begin{array}{l} \text{za } q=0 \text{ dobijamo aritmetički niz} \\ \text{za } d=0 \text{ dobijamo geometrijski niz} \end{array} \right) \end{aligned}$$

Ideja → formirati novi niz koji će biti ili aritmetički ili geometrijski i koji će se od datog niza razlikovati za konstantu; naravno potrebno je proveriti da li je to uopšte moguće izvesti:

Proba: $\begin{aligned} a_n &= b_n + k & a_n &= q'b_n \\ b_{n+1} &= q'b_n & b_{n+1} &= b_n + k \end{aligned}$

Iz \blacklozenge sledi $\begin{cases} b_{n+1} + k = q(b_n + k) + d \\ q'b_n + k = q'b_n + qk + d \end{cases}$, pa je ovo moguće učiniti, pri čemu je q' proizvoljno, a q'

$$k = \frac{d}{1-q} = \text{const}$$

ćemo izabrati tako da je $q' = q$.

Dakle, niz $b_n = a_n - \frac{d}{1-q}$ je geometrijski, pa je njegovo **opšte rešenje** $b_n = Cq^n$

Znači $Cq^n = a_n - \frac{d}{1-q} \Rightarrow a_n = Cq^n + \frac{d}{1-q}$, $q \neq 1$.

ZAKLJUČAK: Opšte rešenje diferencne jednačine $a_{n+1} = q \cdot a_n + d$ je

$$a_n = \begin{cases} C \cdot q^n + \frac{d}{1-q}, & \text{za } q \neq 1 \\ C + nd, & \text{za } q = 1 \text{ (aritm.niz)} \end{cases}$$

PRIMER: $a_1 = 2$ i $a_{n+1} = 7 + 4a_n$, $n \geq 1$

$q=4$, $d=7 \Rightarrow$ opšte rešenje je $a_n = C \cdot 4^n - \frac{7}{3}$, a uz početni uslov $a_1 = 2$ dobijamo $2 = C \cdot 4^1 - \frac{7}{3}$, tj.
 $C = \frac{13}{12}$, pa je $a_n = \frac{13}{12} \cdot 4^n - \frac{7}{3}$ ili $a_n = \frac{13}{3} \cdot 4^{n-1} - \frac{7}{3} = \frac{1}{3}(13 \cdot 4^{n-1} - 7)$.

► OPŠTA LINEARNA DIFERENCNA JEDNAČINA DRUGOG REDA SA KONSTANTNIM KOEFICIJENTIMA

$$a_{n+2} = f(a_{n+1}, a_n)$$

I TIP $a_{n+2} + p \cdot a_{n+1} + q \cdot a_n = 0$; $p, q - const$

Ideja → sniziti red jednačine, tj. svesti je na jednačinu prvog reda, imitirajući postupak rastavljanja na činioce kvadratnog trinoma.

Primer1. $a_{n+2} - 3a_{n+1} + 2a_n = 0 \Rightarrow a_{n+2} - a_{n+1} - 2a_{n+1} + 2a_n = 0 \Rightarrow$

$$\underbrace{(a_{n+2} - a_{n+1})}_{b_{n+1}} - 2\underbrace{(a_{n+1} - a_n)}_{b_n} = 0 \Rightarrow$$

uvodimo smenu $b_{n+1} = a_{n+2} - a_{n+1} \Rightarrow$

$$b_{n+1} = 2b_n \Rightarrow \frac{b_{n+1}}{b_n} = 2 \Rightarrow b_n = K \cdot 2^n \Rightarrow K \cdot 2^n = a_{n+1} - a_n \Rightarrow$$

$$a_{n+1} = a_n + K \cdot 2^n$$

$$a_{n+1} = a_n + K \cdot 2^n = a_{n-1} + K \cdot 2^{n-1} + K \cdot 2^n = a_{n-2} + K \cdot 2^{n-2} + K \cdot 2^{n-1} + K \cdot 2^n = \dots = a_1 + K(2 + 2^2 + \dots + 2^n)$$

$$a_{n+1} = a_1 + K \cdot 2 \cdot \frac{2^{n-1} - 1}{2 - 1} = a_1 + 2K(2^{n-1} - 1) = \underbrace{(a_1 - 2K)}_{M=const} + K \cdot 2^n = M + K \cdot 2^n$$

Znači, opšte rešenje jednačine je $a_{n+1} = M + K \cdot 2^n$

Uz neke početne uslove izračunaju se konstante M i K → npr. za $a_1 = 3$ i $a_2 = 7$ dobijamo

$$\begin{cases} M + 2K = 3 \\ M + 4K = 7 \end{cases} \Rightarrow \begin{cases} K = 2 \\ M = -1 \end{cases} \Rightarrow a_n = 2^{n+1} - 1$$

Primer2. $a_{n+2} - 5a_{n+1} + 6a_n = 0 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x_1 = 2 \vee x_2 = 3 \Rightarrow$

$$a_{n+2} - (2+3)a_{n+1} + 2 \cdot 3a_n = 0 \Rightarrow \underbrace{a_{n+2} - 2a_{n+1}}_{b_{n+1}} - 3(a_{n+1} - 2a_n) = 0 \Rightarrow$$

smena → $b_{n+1} = a_{n+2} - 2a_{n+1} \Rightarrow b_{n+1} - 3b_n = 0 \Rightarrow b_{n+1} = 3b_n \Rightarrow$

$$b_n = K \cdot 3^n$$

Ali zbog koeficijenta 2, sumiranje niza (a_{n+1}) je nešto složenije nego u prethodnom primeru (u kome je jedno od rešenja kvadratne jednačine bilo jednako 1) →

$$a_{n+1} = 2a_n + b_n = 2a_n + K \cdot 3^n = 2(2a_{n-1} + K \cdot 3^{n-1}) + K \cdot 3^n = 2^2 a_{n-1} + K(2 \cdot 3^{n-1} + 3^n) \Rightarrow$$

$$a_{n+1} = 2^2 a_{n-1} + K(2 \cdot 3^{n-1} + 3^n) = 2^2 (2a_{n-2} + K \cdot 3^{n-2}) + K(2 \cdot 3^{n-1} + 3^n) \Rightarrow$$

$$a_{n+1} = 2^3 a_{n-2} + K(2^2 \cdot 3^{n-2} + 2 \cdot 3^{n-1} + 3^n) = \dots = 2^n a_1 + K(2^{n-1} \cdot 3 + 2^{n-2} \cdot 3^2 + \dots + 2 \cdot 3^{n-1} + 3^n) \Rightarrow$$

$$(\Delta) \quad a_{n+1} = 2^n a_1 + 3K(2^{n-1} + 2^{n-2} \cdot 3^1 + \dots + 2 \cdot 3^{n-2} + 3^{n-1})$$

$$a_{n+1} = 2^n a_1 + 3 \cdot 2^{n-1} K \left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \dots + \left(\frac{3}{2}\right)^{n-2} + \left(\frac{3}{2}\right)^{n-1} \right) = 2^n a_1 + 3 \cdot 2^{n-1} K \cdot \frac{\left(\frac{3}{2}\right)^n - 1}{\frac{3}{2} - 1} \Rightarrow$$

$$a_{n+1} = 2^n a_1 + 3 \cdot 2^{n-1} K \cdot \cancel{2}^{n-1} K \cdot \cancel{2}^{n-1} \cancel{K} \cdot \cancel{3}^{n-1} = 2^n a_1 + 3 \cdot K(3^n - 2^n) = \underbrace{(a_1 - 3K)}_M 2^n + \underbrace{3K \cdot 3^n}_L = M \cdot 2^n + L \cdot 3^n$$

Međutim može i ovako

$$\begin{array}{ll} a_2 - 2a_1 = K \cdot 3 \quad / \cdot 2^{n-2} & 2^{n-2} \cancel{a_2} - 2^{n-1} a_1 = K \cdot 3 \cdot 2^{n-2} \\ a_3 - 2a_2 = K \cdot 3^2 \quad / \cdot 2^{n-3} & \cancel{2^{n-3} a_3} - \cancel{2^{n-2} a_2} = K \cdot 3^2 \cdot 2^{n-3} \\ a_4 - 2a_3 = K \cdot 3^3 \quad / \cdot 2^{n-4} & 2^{n-4} a_4 - \cancel{2^{n-3} a_3} = K \cdot 3^3 \cdot 2^{n-4} \\ \dots & \dots \\ a_{n-2} - 2a_{n-3} = K \cdot 3^{n-3} \quad / \cdot 2^2 & \cancel{2^2 a_{n-2}} - \cancel{2^3 a_{n-3}} = K \cdot 3^{n-3} 2^2 \\ a_{n-1} - 2a_{n-2} = K \cdot 3^{n-2} \quad / \cdot 2 & \cancel{2 a_{n-1}} - \cancel{2^2 a_{n-2}} = K \cdot 3^{n-2} 2 \\ a_n - 2a_{n-1} = K \cdot 3^{n-1} & a_n - \cancel{2 a_{n-1}} = K \cdot 3^{n-1} \end{array} \Rightarrow \dots \Rightarrow$$

$$a_n - 2^{n-1} a_1 = 3K(2^{n-2} + 3 \cdot 2^{n-3} + 3^2 \cdot 2^{n-4} + \dots + 3^{n-4} \cdot 2^2 + 3^{n-3} \cdot 2 + 3^{n-2})$$

Dalji postupak izračunavanja je isti kao u (Δ) . Imitirajući ovaj postupak, ali "unazad" do čisto geometrijskog niza dolazimo na sledeći način:

$$a_{n+2} - 5a_{n+1} + 6a_n = 0 \quad \Rightarrow \quad a_{n+2} - 2a_{n+1} - 3(a_{n+1} - 2a_n) = 0$$

$$\text{smena 1.} \rightarrow a_n = 2^n b_n \quad (\text{ili } a_n = 3^n b_n) \quad \Rightarrow$$

$$2^{n+2} b_{n+2} - 2 \cdot 2^{n+1} b_{n+1} - 3(2^{n+1} b_{n+1} - 2 \cdot 2^n b_n) = 0$$

$$2^{n+2}(b_{n+2} - b_{n+1}) - 3 \cdot 2^{n+1}(b_{n+1} - b_n) = 0 \quad / : 2^{n+1} \quad \Rightarrow$$

$$2(b_{n+2} - b_{n+1}) - 3(b_{n+1} - b_n) = 0$$

$$\text{smena 2.} \rightarrow c_{n+1} = b_{n+2} - b_{n+1} \quad \Rightarrow \quad 2c_{n+1} - 3c_n = 0 \quad \Rightarrow \quad c_n = K \cdot \left(\frac{3}{2}\right)^n$$

$$\begin{aligned}
b_2 - b_1 &= K \cdot \frac{3}{2} \\
b_3 - b_2 &= K \cdot \left(\frac{3}{2}\right)^2 \\
b_{n+2} - b_{n+1} &= K \cdot \left(\frac{3}{2}\right)^n \quad \Rightarrow \quad b_4 - b_3 = K \cdot \left(\frac{3}{2}\right)^3 \quad \Rightarrow \\
&\dots
\end{aligned}$$

$$a_n = M \cdot 2^n + L \cdot 3^n \quad (x_1 = 2, x_2 = 3)$$

Uopštavajući ovaj postupak dobijamo opšte rešenje diferencne jednačine drugog reda oblika

$$\alpha_{n+2} + p \cdot \alpha_{n+1} + q \cdot \alpha_n = 0 ; \quad p, q - const$$

Neka su $\alpha, \beta \neq 0$ rešenja **KARAKTERISTIČNE JEDNAČINE** $x^2 + px + q = 0$. Tada je

$$a_{n+2} - (\alpha + \beta)a_{n+1} + \alpha\beta a_n = 0 \quad \Rightarrow \quad a_{n+2} - \alpha a_{n+1} - \beta(a_{n+1} - \alpha a_n) = 0$$

smena 1. \rightarrow (■) $a_n = \alpha^n b_n \quad \Rightarrow \quad \alpha^{n+2} b_{n+2} - \alpha^{n+1} \alpha b_{n+1} - \beta(\alpha^{n+1} b_{n+1} - \alpha^n \alpha b_n) = 0 \quad / : \alpha^{n+1}$

$$\Rightarrow \quad \alpha(b_{n+2} - b_{n+1}) - \beta(b_{n+1} - b_n) = 0$$

smena 2. \rightarrow (▲) $c_{n+1} = b_{n+2} - b_{n+1} \quad \Rightarrow \quad \alpha c_{n+1} - \beta c_n = 0 \quad \Rightarrow \quad \frac{c_{n+1}}{c_n} = \frac{\beta}{\alpha} \quad \Rightarrow$

$$(♣) \boxed{c_n = K \cdot \left(\frac{\beta}{\alpha}\right)^n = \gamma^n}$$

$$\cancel{b_2} - b_1 = K \cdot \gamma$$

$$\cancel{b_3} - \cancel{b_2} = K \cdot \gamma^2$$

$$(\Delta), (\clubsuit) \Rightarrow \cancel{b_4} - \cancel{b_3} = K \cdot \gamma^3 \quad \Rightarrow \quad b_n - b_1 = K \gamma (1 + \gamma + \gamma^2 + \dots + \gamma^{n-2}) \quad \Rightarrow$$

$$b_n - \cancel{b_{n-1}} = K \cdot \gamma^{n-1}$$

$$(i) \quad \gamma \neq 1 \text{ tj. } \alpha \neq \beta \quad \Rightarrow \quad b_n - b_1 = K \gamma \frac{\gamma^{n-1} - 1}{\gamma - 1} \Rightarrow \quad b_n = \underbrace{b_1 - \frac{K\gamma}{\gamma - 1}}_{const} + \underbrace{\frac{K}{\gamma - 1} \gamma^n}_{const} \Rightarrow$$

$$(♦) \boxed{b_n = C + D\gamma^n}$$

$$(\blacksquare), (\spadesuit) \Rightarrow a_n = \alpha^n (C + D\gamma^n) = \alpha^n C + \cancel{\alpha^n} \frac{\beta^n}{\cancel{\alpha^n}} D \Rightarrow$$

$$\boxed{a_n = C\alpha^n + D\beta^n, \quad \alpha \neq \beta}$$

$$(ii) \quad \gamma = 1 \text{ tj. } \alpha = \beta \quad \Rightarrow \quad b_n - b_1 = K(n-1) \quad \Rightarrow \quad b_n = \underbrace{b_1 - K}_{const} + Kn \quad \Rightarrow$$

$$(♥) \boxed{b_n = C + Kn}$$

$$(\blacksquare), (\heartsuit) \Rightarrow a_n = \alpha^n \cdot C + Kn\alpha^n \Rightarrow$$

$$\boxed{a_n = (K + Cn) \cdot \alpha^n, \quad \alpha = \beta}$$